Understanding the PASEP solution

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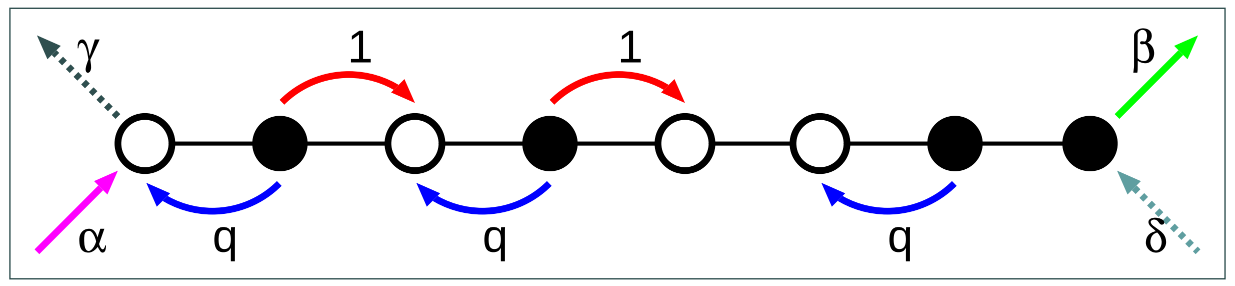
Background

The partially asymmetric exclusion process (PASEP) is a widely studied Markov chain. It has been used as a model in a huge variety of fields, such as a traffic model and as a model for RNA duplication.

The solution involves a normalisation function Z_n. Understanding how Z_n behaves tells us much about behaviour of the process itself.

We analyse Z_n by lattice paths and by patterns and cycles in permutations.

Definition



The PASEP is a string of sites, each of which may be occupied by particles or not. The particles may jump to unoccupied neighbouring sites, and in and out at the ends, as illustrated above.

The process has 6 parameters: α, β, γ, δ, q and the size n.

A central function

It has been shown that the generating function for Z_n can, in the case $\gamma = \delta = 0$, be written as a continued fraction

$$Z_n = [z^n]F(q, 1/\alpha, 1/\beta, z),$$

where

$$F(q,x,y,z) = \frac{1}{1 - z([1]_q^x + [1]_q^y) - \cfrac{z^2[1]_q[2]_q^{x,y}}{1-z([2]_q^x+[2]_q^y)-\cfrac{z^2[2]_q[3]_q^{x,y}}{1-z([3]_q^x+[3]_q^y)\cdots}}}$$

and

$$[h]_q = 1 + q + \cdots + q^{h-2} + q^{h-1}$$

$$[h]_q^x = 1 + q + \cdots + q^{h-2} + xq^{h-1}$$

$$[h]_q^y = 1 + q + \cdots + q^{h-2} + yq^{h-1}$$

$$[h]_q^{x,y} = 1+q+\cdots+q^{h-3}+(x+y-xy)q^{h-2}+xyq^{h-1}.$$

Interpreting F as the generating function for lattice paths enables us to find bijections between these paths and permutations, and describe F as the generating function for permutation statistics.

Interpretations

Firstly $[z^n]F(1,1,1,z) = n!$.

Next, $F(q,1,1,z)$ is the generating function for permutations with respect to length (z) and the number of occurrences of the pattern 2–13.

x is for cycles: $F(1,x,1,z)$ is the generating function for permutations with respect to length (z) and the number of cycles (x).

$F(1,x,y,z)$ counts permutations with respect to length (z) and the number of cycles of which some are marked (x and y).

$F(q,x,1,z)$ is the generating function for permutations with respect to length (z) and the number of cycles (x) and *cyclic* occurrence of the pattern 2–13 (q).

We do not know yet how to fully interpret $F(q,x,y,z)$.

Robert Parviainen is an ARC Research Fellow. His research interests include investigations into various problems in Probability and Combinatorics.